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Stochastic dominance orders of all finite degrees are defined on the set of distribution functions on the nonnegative real numbers in terms of integrals of the distributions. It is proved that if F strictly nth-degree stochastically dominates G, and if the moments of F and G through order n are finite with $\mu_F^k = \int x^k dF(x)$, then $(\mu_F^1,\ldots,\mu_F^n) \neq (\mu_G^1,\ldots,\mu_G^n)$ and $(-1)^{k-1}\mu_F^k > (-1)^{k-1}\mu_G^k$ for the smallest k for which $\mu_F^k \neq \mu_G^k$.

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STOCHASTIC DOMINANCE AND MOMENTS OF DISTRIBUTIONS

Peter C. Fishburn
College of Business Administration
The Pennsylvania State University
University Park, Pennsylvania 16802

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1. Introduction

Stochastic dominance provides a way of analyzing risky decisions when a decision agent's von Neumann-Morgenstern (1947) utility function u is not fully known but is presumed to be in a class U of real-valued functions defined on a consequence space X. When p and q are probability measures on a suitable algebra of subsets of X, we sometimes say that p stochastically dominates q with respect to U if $\int u(x)dp(x) > \int u(x)dq(x)$ for all $u \in U$ for which the expected utilities are finite.

Although stochastic dominance is relevant whenever decision alternatives are described by probability measures, its most popular use involves situations in which X is a set of real numbers with x preferred to y when $x > y$. This context is emphasized in the introduction to stochastic dominance given by Fishburn and Vickson (1978) and will be the context used for the present paper. Definitions of various degrees of stochastic dominance in the real X setting can be given without explicit mention of U classes, and I will follow this approach here.

My purpose is to establish a general connection between nth-degree stochastic dominance and the first n moments of probability measures whose supports are bounded below, for all $n \in \{1,2,...\}$. It will be assumed that the measures are countably additive and that their supports are bounded below by 0. The corresponding distribution functions F,G,... will be taken to be continuous from the right with $F(x) = 0$ for all $x < 0$ and $F(x) \to 1$ as $x \to \infty$.

The basic definitions and main theorem are presented and discussed in the next section where the main theorem is observed to follow from two auxiliary theorems. Proofs of the latter theorems are given in the final two sections. These proofs are based solely on the distribution functions and will not involve utility classes.

2. Definitions and Theorems

Let $\mathcal{F}$ be the set of all right-continuous distribution functions on the real line with $F(x) = 0$ for all $x < 0$ for each $F \in \mathcal{F}$. For each $F \in \mathcal{F}$ let $F^1 = F$ and recursively define

$$F^{n+1}(x) = \int_0^x F^n(y)dy \quad \text{for all } x \geq 0 \text{ and } n \in \{1,2,\ldots\}.$$

Nonstrict ($\geq_n$) and strict ($>_n$) nth-degree stochastic dominance relations are then defined on $\mathcal{F}$ as follows:

$$F \geq_n G \text{ iff } F^n(x) \leq G^n(x) \quad \text{for all } x \in [0,\infty),$$

$$F >_n G \text{ iff } F \neq G \text{ and } F \geq_n G.$$

These relations are transitive and increasingly more inclusive: $\geq_1 \subset \geq_2 \subset \ldots$ and $>_1 \subset >_2 \subset \ldots$ where $\subset$ denotes proper inclusion. Utility classes that are congruent with the first few stochastic dominance orders are presented in Fishburn and Vickson (1978, pp. 102-113). For example, $F \geq_1 G$ iff $\int u(x)dF(x) \geq \int u(x)dG(x)$ for all nondecreasing u on $[0,\infty)$ for which the expectations exist, and $F \geq_2 G$ iff $\int udF \geq \int udG$ for all nondecreasing and concave u on $[0,\infty)$ for which the expectations exist. The corresponding theorems for $(>_1, >)$ and $(>_2, >)$ respectively involve strictly increasing u and strictly increasing-strictly concave u. Congruent utility classes for the third-degree relations are the subsets of the second-degree classes in which first derivatives exist and are convex ($\geq_3$) or strictly convex ($>_3$). Although it is traditional to define $F \geq_3 G$ iff $F^3(x) \leq G^3(x)$ for all x <u>and</u> the mean of F is as great as the mean of G (e.g., [Whitmore, 1970], [Fishburn and Vickson, 1978]), we shall see that the condition on the means is redundant in the present formulation.

The moment sequence for $F \in \mathcal{F}$ is $\mu(F) = (\mu_F^0, \mu_F^1, \mu_F^2, \ldots)$ where $\mu_F^0 = \int dF(x) = 1$ and $\mu_F^n = \int x^n dF(x)$ for each $n \geq 1$. The part of this sequence through the nth moment will be denoted as $\mu_n(F) = (\mu_F^0, \ldots, \mu_F^n)$. Our main theorem will be based on binary relations $>_n^*$ on $\{\mu(F)\colon\ F \in \mathcal{F}\}$ defined lexicographically on the basis of μ_n as follows:

$$\mu(F) >_n^* \mu(G) \text{ iff } \mu_n(F) \neq \mu_n(G) \text{ and } (-1)^{k-1}\mu_F^k > (-1)^{k-1}\,\mu_G^k$$

$$\text{for the smallest } k \text{ for which } \mu_F^k \neq \mu_G^k.$$

For example, $\mu(F) >_1^* \mu(G)$ iff $\mu_F^1 > \mu_G^1$, and $\mu(F) >_2^* \mu(G)$ iff either $\mu_F^1 > \mu_G^1$ or $(\mu_F^1 = \mu_G^1,\ \mu_F^2 < \mu_G^2)$. Alternatively, $\mu(F) >_2^* \mu(G)$ iff either F has the greater mean or the means of F and G are equal and F has a smaller variance than G.

Theorem 1. _If_ $F, G \in \mathcal{F}$ _and the moments of_ F _and_ G _through order_ n _are finite_, _then_ $\mu(F) >_n^* \mu(G)$ _if_ $F >_n G$.

For discussion purposes the next three paragraphs assume that the moments involved therein are finite.

The $n = 1$ part of Theorem 1 says that if $F >_1 G$ then the mean of F must be greater than the mean of G. Equivalently, if $\mu_F^1 \leq \mu_G^1$ then F cannot strictly first-degree stochastically dominate G. This fact seems to be widely known.

It appears to be less commonly recognized that $F >_2 G$ implies that either $\mu_F^1 > \mu_G^1$ (in which case the inequality on second moments could go either way) or $(\mu_F^1 = \mu_G^1,\ \mu_F^2 < \mu_G^2)$. Parts of this result have been noted by Hanoch and Levy (1969), Rothschild and Stiglitz (1970) and Fishburn and Vickson (1978, p. 78).

For n = 3, Theorem 1 says that $F >_3 G$ entails either $\mu_F^1 > \mu_G^1$ or $(\mu_F^1 = \mu_G^1, \mu_F^2 < \mu_G^2)$ or $(\mu_F^1 = \mu_G^1, \mu_F^2 = \mu_G^2, \mu_F^3 > \mu_G^3)$. Although parts of this have been noted by Whitmore (1970) and Fishburn and Vickson (1978, pp. 78-82) for special cases, the complete result appears to be previously unknown. In like manner, Theorem 1 for $n \geq 4$ appears to be new.

Since $F >_3 G$ implies $\mu_F^1 \geq \mu_G^1$ when these means are finite--and even when one or both are infinite--our definition of $>_3$ (or $\geq_3$) is formally equivalent to the traditional definition. It should be noted however that upper-bounded definitions are not so simply related. In particular, if [0,b] with b finite includes the supports of F and G then, as shown in Fishburn (1976), it is possible to have $\mu_F^1 < \mu_G^1$ when $F^3(x) \leq G^3(x)$ for all $x \in [0,b]$. In this case, when $F^3(x) \leq G^3(x)$ is specified only over an interval that includes the supports of F and G, it is necessary to include the stipulation $\mu_F^1 \geq \mu_G^1$ in the definition of $F \geq_3 G$ if we want $\geq_3$ to be congruent with the type of U class mentioned in the opening paragraph of this section.

The proof of Theorem 1 can be based on the following two auxiliary theorems that will be proved in the next two sections.

Theorem 2. _If_ $F,G \in \mathcal{F}$, _if_ $F \geq_m G$ _for some_ $m \in \{1,2,\ldots\}$, _and if the moments of_ F _and_ G _through order_ $n + 1 \geq 1$ _are finite with_ $\mu_F^k = \mu_G^k$ _for_ $k = 0,1,\ldots,n$, _then_ $(-1)^n \mu_F^{n+1} \geq (-1)^n \mu_G^{n+1}$.

Theorem 3. _If_ $F,G \in \mathcal{F}$, _if_ $F >_n G$, _and if the moments of_ F _and_ G _through order_ $n \geq 1$ _are finite with_ $\mu_F^k = \mu_G^k$ _for_ $k = 0,1,\ldots,n-1$, _then_ $(-1)^{n-1}\mu_F^n > (-1)^{n-1} \mu_G^n$.

The latter theorem implies that $\mu_n(F) = \mu_n(G)$ cannot be true when $F >_n G$, and the former theorem says that if F stochastically dominates G in the nonstrict sense for any finite degree and if $\mu_n(F) = \mu_n(G)$ then $\mu_F^{n+1} \geq \mu_G^{n+1}$ if n is even and $\mu_F^{n+1} \leq \mu_G^{n+1}$ if n is odd, provided that the moments involved are finite. Theorem 1 follows immediately from Theorems 2 and 3: if $F >_n G$ then $F \geq_n G$, hence either $\mu(F) >_n^* \mu(G)$ or $\mu_n(F) = \mu_n(G)$ by Theorem 2; since Theorem 3 rules out $\mu_n(F) = \mu_n(G)$ we are left with $\mu(F) >_n^* \mu(G)$.

3. Proof of Theorem 2

We begin with a lemma that leads to the proper sense of the inequality in the conclusion of Theorem 2.

Lemma 1. For any $H \in \mathcal{F}$ with finite moments through order n and for all $m > n \geq 0$ and $x > 0$ let

$$T_{n,m}(H,x) = \sum_{k=0}^{n} (-1)^{k+1}\binom{m}{k}\mu_H^k/x^k + \int_{y=0}^{x} (1 - y/x)^m dH(y). \tag{1}$$

Then $T_{n,m}(H,x) = 0$ iff $H(0) = 1$ and, for all other $H \in \mathcal{F}$, $T_{n,m}(H,x) > 0$ if n is odd, and $T_{n,m}(H,x) < 0$ if n is even.

Proof. If H assigns probability mass 1 to $y = 0$ then (1) gives $T_{n,m}(H,x) = -1 + 1 = 0$. Assume henceforth that $H(0) < 1$. When μ_H^k in the right hand side of (1) is replaced by $\int_0^x y^k dH(y) + \int_x^\infty y^k dH(y)$, and $(1 - y/x)^m$ is expanded binomially, we get

$$\begin{aligned} T_{n,m}(H,x) &= \int_0^x \Big\{ \sum_{k=n+1}^{m} (-1)^k\binom{m}{k}(y/x)^k\Big\} dH(y) + \int_x^\infty \Big\{ \sum_{k=0}^{n} (-1)^{k+1}\binom{m}{k}(y/x)^k\Big\} dH(y) \\ &= \int_0^x A_{n,m}(z)dH(y) + \int_x^\infty B_{n,m}(z)dH(y) \end{aligned} \tag{2}$$

where the A and B terms are defined in context as the sums under the integrals and $z = y/x$ with $z \in [0,1]$ for A and $z \in [1,\infty)$ for B. We consider A and B in turn.

First, $A_{n,m}(0) = 0$ for all $m > n \geq 0$ and, since $A_{0,m}(z) = (1-z)^m - 1$, $A_{0,m}(z) < 0$ for all $z \in (0,1]$. Since

$$A'_{n,m}(z) = dA_{n,m}(z)/dz = -mA_{n-1,m-1}(z) \text{ for } m > n \geq 1 \text{ and } z > 0,$$

it follows that $A'_{1,m}(z) > 0$ for $z > 0$, hence by continuity and $A_{1,m}(0) = 0$ that $A_{1,m}(z) > 0$ for all $z \in (0,1]$ and $m > 1$. Then $A'_{2,m}(z) < 0$ for $z > 0$ and $m > 2$, so that $A_{2,m}(z) < 0$ for all $z \in (0,1]$. The obvious continuation of this process gives

$$A_{n,m}(z) > 0 \text{ for all } z \in (0,1] \text{ if n is odd,}$$

$$A_{n,m}(z) < 0 \text{ for all } z \in (0,1] \text{ if n is even,}$$

along with $A_{n,m}(0) = 0$.

For the B part we note first (e.g. [Feller, 1957, p. 61]) that $B_{n,m}(1) = (-1)^{n+1}\binom{m-1}{n}$ so that $B_{n,m}(1) > 0$ if n is odd, and $B_{n,m}(1) < 0$ if n is even. In addition,

$$B'_{n,m}(z) = -m\,B_{n-1,m-1}(z) \text{ for } m > n \geq 1 \text{ and } z \geq 1.$$

Since $B_{0,m}(z) \equiv -1$, this implies that $B'_{1,m}(z) > 0$ for $m > 1$ and $z \geq 1$, hence that $B_{1,m}(z) > 0$ for all $z \geq 1$ since $B_{1,m}(1) > 0$. Then $B'_{2,m}(z) < 0$, hence $B_{2,m}(z) < 0$ for all $z \geq 1$ since $B_{2,m}(1) < 0$. The continuation of this process shows that

$$B_{n,m}(z) > 0 \text{ for all } z \in [1,\infty) \text{ if } n \text{ is odd,}$$

$$B_{n,m}(z) < 0 \text{ for all } z \in [1,\infty) \text{ if } n \text{ is even.}$$

These conclusions along with those for A and the hypothesis that H has positive probability mass on $x > 0$ then yield the final conclusions of Lemma 1 in view of (2). Q.E.D.

For Theorem 2, suppose that $F,G \in \mathcal{F}$ have finite moments through order $n + 1 \geq 1$ and that $\mu_n(F) = \mu_n(G)$, i.e. $\mu_F^k = \mu_G^k$ for all $k \leq n$. Assume also that $F \geq_m G$ for some m. Since $\geq_m \subset \geq_{m+1}$ for all m, select $m > n$ with $F \geq_{m+1} G$. Then, by the definition of $\geq_{m+1}$ and Fishburn (1976),

$$\int_0^x (x - y)^m dF(y) \leq \int_0^x (x - y)^m dG(y) \text{ for all } x > 0.$$

Since $\mu_n(F) = \mu_n(G)$, it follows that

$$T_{n,m}(G,x) \geq T_{n,m}(F,x) \text{ for all } x > 0. \tag{3}$$

Suppose first that $G(0) = 1$ so that $T_{n,m}(G,x) = 0$ by Lemma 1. Then $0 \geq T_{n,m}(F,x)$ for all $x > 0$. It follows that either $F(0) = 1$ and $F = G$, in which case $\mu_F^{n+1} = \mu_G^{n+1} = 0$, or that $F(0) < 1$, in which case Lemma 1 implies that $0 > T_{n,m}(F,x)$ for $x > 0$ and hence that n is even. But with n even, $(-1)^n \mu_F^{n+1} > (-1)^n \mu_G^{n+1} = 0$ in the latter case. Hence the conclusion of Theorem 2 holds if $G(0) = 1$.

Assume henceforth that $G(0) < 1$, in which case $F(0) < 1$ also. Then (3) applies with neither term ever equal to zero according to the last part of Lemma 1. We shall prove that

$$\lim_{x\to\infty} \frac{T_{n,m}(F,x)}{T_{n,m}(G,x)} = \frac{\mu_F^{n+1}}{\mu_G^{n+1}} \tag{4}$$

which, in view of (3) and the final conclusions of the lemma, implies that $(-1)^n \mu_F^{n+1} \geq (-1)^n \mu_G^{n+1}$, as desired for Theorem 2. Since $\lim T_{n,m}(H,x) = -\mu_H^0 + 1 = 0$, we differentiate the numerator and denominator of the left hand side of (4) with respect to x in anticipation of using l'Hospital's rule. When m/x^2 is canceled from the numerator and denominator after the differentiations have been completed, we get

$$\frac{d\, T_{n,m}(F,x)/dx}{d\, T_{n,m}(G,x)/dx} = \frac{\sum_{k=0}^{n-1} (-1)^{k+1}\binom{m-1}{k}\mu_F^{k+1}/x^k + \int_0^x (1 - y/x)^{m-1} y\,dF(y)}{\sum_{k=0}^{n-1} (-1)^{k+1}\binom{m-1}{k}\mu_G^{k+1}/x^k + \int_0^x (1 - y/x)^{m-1} y\,dG(y)} . \tag{5}$$

If n = 0 then the numerator and denominator of (5) respectively approach μ_F^1 and μ_G^1 as $x \to \infty$ by the monotone convergence theorem (e.g., [Loeve, 1960, p. 124]), and a single application of l'Hospital's rule shows that (4) holds. If $n > 1$, then the numerator and denominator of the right hand side of (5) both go to zero ($-\mu_F^1 + \mu_F^1$ and $-\mu_G^1 + \mu_G^1$) as $n \to \infty$. In this case methods similar to those used for Lemma 1 show that both parts of the right hand side of (5) have constant nonzero sign. We then differentiate the numerator and denominator of the right hand side of (5) and cancel $(m - 1)/x^2$ to get

$$\frac{\sum_{k=0}^{n-2} (-1)^{k+1}\binom{m-2}{k}\mu_F^{k+2}/x^k + \int_0^x (1 - y/x)^{m-2} y^2\,dF(y)}{\sum_{k=0}^{n-2} (-1)^{k+1}\binom{m-2}{k}\mu_G^{k+2}/x^k + \int_0^x (1 - y/x)^{m-2} y^2\,dG(y)} .$$

If n = 1 then (4) holds by two applications of l'Hospital's rule since the limit of the expression just written is μ_F^2/μ_G^2. If $n > 1$, we continue in the indicated manner until we arrive at the ratio $\int_0^x (1 - y/x)^{m-n-1} y^{n+1} dF(y) / \int_0^x (1 - y/x)^{m-n-1} y^{n+1}$ $dG(y)$, whose limit is μ_F^{n+1}/μ_G^{n+1}. Hence n + 1 applications of l'Hospital's rule yield a general verification of (4).

4. Proof of Theorem 3

Throughout this section we use the convention that $t^0 = 1$ when $t = 0$. In anticipation of using the monotone convergence theorem later we shall first prove

Lemma 2. *Suppose* $H \in \mathcal{F}$ *has finite moments through order* $n - 1$ *with* $n \geq 1$, *and* $0 \leq y \leq x$ *with* $x > 0$. *Then*

$$
\begin{aligned}
&(-1)^n [n! H^n(y) + \sum_{k=1}^{n} (-1)^{n-k+1} \binom{n}{k} k \mu_H^{n-k} y^{k-1}] \\
&\geq (-1)^n [n! H^n(y) + \sum_{k=1}^{n} (-1)^{n-k+1} \binom{n}{k} k y^{k-1} \int_{z=0}^{x} z^{n-k} dH(z)] \\
&\geq 0.
\end{aligned}
$$

Proof. We note first that

$$
\sum_{k=1}^{n} (-1)^{n-k+1} \binom{n}{k} k \mu_H^{n-k} y^{k-1} = n \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} y^k \mu_H^{n-k-1},
$$

$$
\begin{aligned}
\sum_{k=1}^{n} (-1)^{n-k+1} \binom{n}{k} k y^{k-1} \int_{z=0}^{x} z^{n-k}\, dH(z) &= n \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} y^k \int_{z=0}^{x} z^{n-k-1} dH(z) \\
&= -n \int_{z=0}^{x} (y - z)^{n-1} dH(z),
\end{aligned}
$$

$$
n! H^n(y) = n \int_{z=0}^{y} (y - z)^{n-1} dH(z) \qquad \text{([Fishburn, 1976]).}
$$

By the first two of these, the first inequality in the conclusion of Lemma 2 holds if and only if

$$n(-1)^n\left[\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n-1}{k}y^k\left\{\mu_H^{n-k-1} - \int_{z=0}^{x} z^{n-k-1}dH(z)\right\}\right] \geq 0,$$

which is true if and only if

$$\sum_{k=0}^{n-1} (-1)^k\binom{n-1}{k}y^k \int_{z=x}^{\infty} z^{n-k-1}dH(z) \geq 0.$$

But this is true since its left hand side equals $\int_{z=x}^{\infty} (z - y)^{n-1}dH(z)$. By the final two expressions in the opening sentence of this proof, the second ($\geq$ 0) inequality in Lemma 2 holds if

$$(-1)^n[-n \int_{z=y}^{x} (y - z)^{n-1}dH(z)] \geq 0,$$

which is true since its left hand side equals $n \int_{z=y}^{x} (z - y)^{n-1}dH(z)$. Q.E.D.

When dy is replaced by $e^y d(1 - e^{-y})$ in the following expressions, with $1 - e^{-y}$ a distribution function in F, Lemma 2 in conjunction with the monotone convergence theorem shows that if μ_H^{n-1} is finite then

$$(-1)^n \int_{y=0}^{\infty} [n!H^n(y) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n}{k}k\mu_H^{n-k}y^{k-1}]dy$$

$$= \lim_{x\to\infty} \{(-1)^n \int_{y=0}^{x} [n!H^n(y) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n}{k}ky^{k-1} \int_{z=0}^{x} z^{n-k}dH(z)]dy\}.$$

The expression within braces after lim reduces as follows:

$$(-1)^n[n!H^{n+1}(x) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n}{k}x^k \int_{z=0}^{x} z^{n-k}dH(z)]$$

$$= (-1)^n\left[\int_{y=0}^{x} (x - y)^n dH(y) + \sum_{k=1}^{n} (-1)^{n-k+1}\binom{n}{k}x^k \int_{y=0}^{x} y^{n-k}dH(y)\right]$$

$$= (-1)^n \left[\int_{y=0}^{x} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k y^{n-k} dH(y) + \sum_{k=1}^{n} (-1)^{n-k+1} \binom{n}{k} x^k \int_{y=0}^{x} y^{n-k} dH(y) \right]$$

$$= \int_{y=0}^{x} y^n dH(y).$$

Therefore, when μ_H^{n-1} is finite,

$$\int_{y=0}^{\infty} [n!H^n(y) + \sum_{k=1}^{n} (-1)^{n-k+1} \binom{n}{k} k\mu_H^{n-k} y^{k-1}]dy = (-1)^n \mu_H^n. \tag{6}$$

Finally, as in the hypotheses of Theorem 3, suppose that $F >_n G$ and that the moments of F and G through order n are finite with $\mu_F^k = \mu_G^k$ for all $k \leq n - 1$. Then, since $F^n(y) \leq G^n(y)$ for all y, and $F^n(y) < G^n(y)$ in a non-degenerate interval within $[0,\infty)$, (6) shows immediately that $(-1)^n \mu_F^n < (-1)^n \mu_G^n$.

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